QUIZ 19 SOLUTIONS: LESSON 26
NOVEMBER 3, 2017

Write legibly, clearly indicate the question you are answering, and put a box or circle around your final answer. If you do not clearly indicate the question numbers, I will take off points. Write as much work as you need to demonstrate to me that you understand the concepts involved. If you have any questions, raise your hand and I will come over to you.

The following questions will lead you through the Lagrange multipliers method of finding the shortest distance between the point $(-1, 1)$ and the unit circle (the **unit circle** is the circle centered at the origin of radius 1).

Be sure to read each question carefully and mark clearly which question you are answering. Keep your answers exact.

1. [1 pt] Let $f(x, y) = (x+1)^2 + (y-1)^2$ which is the **square** of the distance from any point (x, y) to $(-1, 1)$. If our constraint is that we only want to consider points on the unit circle, then what is $g(x, y) = C$?

   Solution: $g(x, y) = \boxed{x^2 + y^2 = 1}$

2. [1 pt] Complete the following system of equations for $f(x, y)$ and $g(x, y) = C$ from # 1:

$$2(x+1) = \lambda \boxed{}$$

$$2(y-1) = \lambda \boxed{}$$

$$\boxed{} = \boxed{}$$

   Solution:

$$2(x+1) = \lambda \boxed{2x}$$

$$2(y-1) = \lambda \boxed{2y}$$

$$\boxed{x^2 + y^2 = 1}$$

3. [3 pts] Find all the solutions to the system from # 2.

   Solution: From the first equation, we have

$$2(x+1) = 2\lambda x \Rightarrow x + 1 = \lambda x.$$

   We subtract λx and 1 from both sides to get

$$x - \lambda x = -1.$$

This means that $x(1-\lambda) = -1$. Because this is equal to a non-zero number, $x \neq 0$. Thus, $1 - \lambda = -\dfrac{1}{x}$.

From the second equation, we have

$$2(y-1) = 2\lambda y \Rightarrow y - 1 = \lambda y.$$

We subtract λy from both sides and add 1 to both sides to get

$$y - \lambda y = 1 \Rightarrow y(1-\lambda) = 1.$$

Because this is equal to a non-zero number, $y \neq 0$ and so we have $1 - \lambda = \dfrac{1}{y}$.

Thus, we see that

$$-\frac{1}{x} = 1 - \lambda = \frac{1}{y}.$$

Cross-multiplying, we get that $-y = x$. Plugging this into the constraint, we find

$$1 = (-y)^2 + y^2 = 2y^2 \Rightarrow y^2 = \frac{1}{2}.$$

Hence, $y = \pm\dfrac{1}{\sqrt{2}} = \pm\dfrac{\sqrt{2}}{2}$ and we get two solutions

$$\boxed{\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) \quad \text{and} \quad \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)}.$$

4. [3 pts] Evaluate $f(x, y)$ at all of the solutions from # 3.

Solution: $f(x, y) = (x+1)^2 + (y-1)^2$ so

$$\begin{aligned} f\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) &= \left(\frac{\sqrt{2}}{2} + 1\right)^2 + \left(-\frac{\sqrt{2}}{2} - 1\right)^2 \\ &= \left(\frac{\sqrt{2}+2}{2}\right)^2 + \left(\frac{-\sqrt{2}-2}{2}\right)^2 \\ &= 2\left(\frac{(\sqrt{2}+2)^2}{4}\right) = \frac{(\sqrt{2}+2)^2}{2} \longleftarrow \text{max} \end{aligned}$$

$$\begin{aligned} f\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) &= \left(-\frac{\sqrt{2}}{2} + 1\right)^2 + \left(\frac{\sqrt{2}}{2} - 1\right)^2 \\ &= \left(\frac{2-\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}-2}{2}\right)^2 \\ &= 2\left(\frac{(2-\sqrt{2})^2}{4}\right) = \frac{(2-\sqrt{2})^2}{2} \longleftarrow \text{min} \end{aligned}$$

5. [2 pts] What is the minimum distance from $(1, -1)$ to the unit circle?

HINT: Go back to # 1 and read what $f(x, y)$ stands for.

Solution: The minimum distance is the square root of the smaller of the two values above because $f(x, y)$ was the *square* of the distance. Thus, the minimum distance is

$$\sqrt{\frac{(2-\sqrt{2})^2}{2}} = \frac{2-\sqrt{2}}{\sqrt{2}} = \boxed{\sqrt{2}-1} \approx .41421.$$