## QUIZ 19 SOLUTIONS: LESSON 26 NOVEMBER 3, 2017

Write legibly, clearly indicate the question you are answering, and put a box or circle around your final answer. If you do not clearly indicate the question numbers, I will take off points. Write as much work as you need to demonstrate to me that you understand the concepts involved. If you have any questions, raise your hand and I will come over to you.

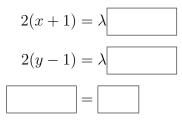
The following questions will lead you through the Lagrange multipliers method of finding the shortest distance between the point (-1, 1) and the unit circle (the **unit circle** is the circle centered at the origin of radius 1).

Be sure to read each question carefully and mark clearly which question you are answering. Keep your answers exact.

1. [1 pt] Let  $f(x, y) = (x+1)^2 + (y-1)^2$  which is the **square** of the distance from any point (x, y) to (-1, 1). If our constraint is that we only want to consider points on the unit circle, then what is g(x, y) = C?

**Solution**:  $g(x,y) = x^2 + y^2 = 1$ 

2. [1 pt] Complete the following system of equations for f(x, y) and g(x, y) = C from # 1:



Solution:

$$2(x+1) = \lambda \boxed{2x}$$
$$2(y-1) = \lambda \boxed{2y}$$
$$\boxed{x^2 + y^2 = 1}$$

3. [3 pts] Find all the solutions to the system from # 2.

**Solution**: From the first equation, we have

$$2(x+1) = 2\lambda x \Rightarrow x+1 = \lambda x.$$

We subtract  $\lambda x$  and 1 from both sides to get

$$x - \lambda x = -1$$

This means that  $x(1 - \lambda) = -1$ . Because this is equal to a non-zero number,  $x \neq 0$ . Thus,  $1 - \lambda = -\frac{1}{x}$ .

From the second equation, we have

$$2(y-1) = 2\lambda y \Rightarrow y - 1 = \lambda y$$

We subtract  $\lambda y$  from both sides and add 1 to both sides to get

$$y - \lambda y = 1 \Rightarrow y(1 - \lambda) = 1$$

Because this is equal to a non-zero number,  $y \neq 0$  and so we have  $1 - \lambda = \frac{1}{y}$ .

Thus, we see that

$$-\frac{1}{x} = 1 - \lambda = \frac{1}{y}.$$

Cross-multiplying, we get that -y = x. Plugging this into the constraint, we find

$$1 = (-y)^2 + y^2 = 2y^2 \Rightarrow y^2 = \frac{1}{2}.$$

Hence,  $y = \pm \frac{1}{\sqrt{2}} = \pm \frac{\sqrt{2}}{2}$  and we get two solutions

$\left(\frac{1}{2}, -\frac{1}{2}\right)$ and $\left(-\frac{1}{2}, \frac{1}{2}\right)$	$\left  \left( \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right) \right $	and	$\left(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$
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4. [3 pts] Evaluate f(x, y) at all of the solutions from # 3. <u>Solution</u>:  $f(x, y) = (x + 1)^2 + (y - 1)^2$  so

$$f\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = \left(\frac{\sqrt{2}}{2} + 1\right)^2 + \left(-\frac{\sqrt{2}}{2} - 1\right)^2$$
$$= \left(\frac{\sqrt{2} + 2}{2}\right)^2 + \left(\frac{-\sqrt{2} - 2}{2}\right)^2$$
$$= 2\left(\frac{(\sqrt{2} + 2)^2}{4}\right) = \frac{(\sqrt{2} + 2)^2}{2} \longleftarrow \max$$
$$f\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \left(-\frac{\sqrt{2}}{2} + 1\right)^2 + \left(\frac{\sqrt{2}}{2} - 1\right)^2$$
$$= \left(\frac{2 - \sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2} - 2}{2}\right)^2$$
$$= 2\left(\frac{(2 - \sqrt{2})^2}{4}\right) = \frac{(2 - \sqrt{2})^2}{2} \longleftarrow \min$$

5. [2 pts] What is the minimum distance from (1, -1) to the unit circle?

**HINT**: Go back to # 1 and read what f(x, y) stands for.

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<u>Solution</u>: The minimum distance is the square root of the smaller of the two values above because f(x, y) was the square of the distance. Thus, the minimum distance is

$$\sqrt{\frac{(2-\sqrt{2})^2}{2}} = \frac{2-\sqrt{2}}{\sqrt{2}} = \boxed{\sqrt{2}-1} \approx .41421.$$